Malaysian Journal of Mathematical Sciences 6(S): 139-150 (2012)
Special Edition of International Workshop on Mathematical Analysis (IWOMA)

Finite Difference Method for the Estimation of a Heat Source Dependent on Time Variable

[1,2]Allaberen Ashyralyev and [1]Abdullah Said Erdogan

[1]Department of Mathematics, Fatih University, Istanbul, Turkey

[2]Department of Mathematics, ITTU, Ashgabat, Turkmenistan

E-mail: aashyr@fatih.edu.tr and aserdogan@fatih.edu.tr

ABSTRACT

Well-posedness of difference scheme for the inverse problem of reconstructing the right side of a parabolic equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = a(x)\dfrac{\partial^2 u(t,x)}{\partial x^2} - \sigma u(t,x) + p(t)q(x) + f(t,x), \\ 0 < x < 1, \; 0 < t \le T, \\ u_x(t,0) = 0, \; u_x(t,l) = \psi(t), \; 0 \le t \le T, \\ u(0,x) = \varphi(x), \; 0 \le x \le l, \\ u(t,x^*) = \rho(t), \; 0 \le x^* \le l, \; 0 \le t \le T, \end{cases}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $f(t,x)$, $q(x)$, $\varphi(x)$, $\psi(t)$ and $\rho(t)$ are given functions, $a(x) \ge \delta > 0$ and $\sigma > 0$ is a sufficiently large number. Numerical methods for estimation of constant terms of coercive stability estimates are described.

2000 MSC: 65N12, 65M12, 65J10

Keywords: Parabolic inverse problem, well-posedness, difference scheme.

1. INTRODUCTION

The inverse problem of reconstructing the right hand side of a parabolic equation has been considered in many papers (see Borukhov and Vabishchevich (2000), Samarskii and Vabishchevich (2007) and the references therein). The inverse heat conduction problems deal with the determination of the crucial parameters in analysis such as determination of boundary conditions, the internal energy source, thermal conductivity, the volumetric heat capacity, etc. They have been widely applied in many designs and manufacturing problems especially in which direct measurements of surface conditions are not possible. The formulation of numerical methods and literature review is given by many researchers. In order to determine unknown conditions, these methods have often been combined with the optimization algorithms such as regularization technique.

The theoretical statements on well-posedness of the inverse problem with one variable has been considered in many theoretical papers (Ivanchov (1995), Choulli and Yamamoto (1996, 1999) and Ashyralyev (2010)). The generic well-posedness of a linear inverse problem is studied for values of a diffusion parameter and generic local well-posedness of an inverse problem is proved in Choulli and Yamamoto (1996, 1999) where the unknown control function is in space variable. In Borukhov and Vabishchevich (2000) and Samarskii and Vabishchevich (2007), the well-posedness of the algorithm for the numerical solution of the identification problem with time variable is investigated in maximum norm. In Ashyralyev (2010), the well-posedness of problem of determining the parameter of a parabolic equation is considered in Hölder spaces.

A homogenous plate with l thickness and constant thermal properties with insulated boundaries heated by a plane surface heat source of $p(t)$ located at a specified position $x = x^*$ can be formulated as a parabolic equation (Liu (2008) and Yang (1998)). Also, in the process of transportation, diffusion and conduction of natural materials, the following heat equation is induced (Yan *et al.* (2008))

$$u_t - a^2 \Delta u = f(t, x; u), \ (t, x) \in (o, t_{\max}] \times \Omega,$$

where u represents state variable, a is the diffusion coefficient, Ω is a bounded domain in $\mathbb{R}^d$ and f denotes physical laws, which means source terms here.

2. FIRST ORDER OF ACCURACY DIFFERENCE SCHEMES AND THE WELL-POSEDNESS

We consider the inverse problem of reconstructing the right side of a parabolic equation with nonlocal conditions

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = a(x)\dfrac{\partial^2 u(t,x)}{\partial x^2} - \sigma u(t,x) + p(t)q(x) + f(t,x), \\ 0 < x < 1,\ 0 < t \le T, \\ u_x(t,0) = 0,\ u_x(t,l) = \psi(t),\ 0 \le t \le T, \\ u(0,x) = \varphi(x),\ 0 \le x \le l, \\ u(t,x^*) = \rho(t),\ 0 \le x^* \le l,\ 0 \le t \le T, \end{cases} \tag{1}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $f(t,x)$, $q(x)$, $\varphi(x)$, $\psi(t)$ and $\rho(t)$ are given functions, $a(x) \ge \delta > 0$ and $\sigma > 0$ is a sufficiently large number. Here x^* is the interior location of a thermocouple recording the temperature measurement. Assume that

(a) $q(x)$ is a sufficiently smooth function,
(b) $q'(0) = q'(l) = 0$,
(c) $q'(x^*) \ne 0$.

The first order of accuracy difference scheme for the approximate solution of the problem (1)

$$\begin{cases} \dfrac{u_n^k - u_n^{k-1}}{\tau} = a(x_n)\dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} - \sigma u_n^k + p^k q_n + f(t_k, x_n), \\ p^k = p(t_k),\ q_n = q(x_n),\ x_n = nh,\ t_k = k\tau, \\ 1 \le k \le N,\ 1 \le n \le M-1,\ Mh = l,\ N\tau = T, \\ u_1^k - u_0^k = 0,\ u_M^k - u_{M-1}^k = h\psi(t_k),\ 0 \le k \le N, \\ u_n^0 = \varphi(x_n),\ 0 \le n \le M, \\ u_{[\frac{x^*}{h}]}^k = u_s^k = \rho(t_k),\ 0 \le k \le N,\ 0 \le s \le M \end{cases} \tag{2}$$

here $q_s \ne 0$, $q_1 = q_0$ and $q_M = q_{M-1}$ are assumed is constructed.

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}_h^{\alpha} = \overset{\circ}{C}{}^{\alpha}[0,l]_h,\ \alpha \in (0,1)$ of all grid functions $\phi^h = \{\phi_n\}_{n=1}^{M-1}$ defined on

$$[0,l]_h = \{x_n = nh,\ 0 \le n \le M,\ Mh = l\}$$

with $\phi_0 = \phi_M$ equipped with the norm

$$\|\phi_h\|_{\overset{\circ}{C}{}_h^{\alpha}} = \max_{1\le n\le M} |\phi_n| + \sup_{1\le n+r\le M} |\phi_{n+r} - \phi_n| (rh)^{-\alpha}.$$

$C_\tau(E) = C([o,T]_\tau, E)$ is the Banach space of all grid functions $\phi^\tau = \{\phi(t,k)\}_{k=1}^{N-1}$ defined on $[0,T]_\tau\{t_k = t_\tau,\ 0 \le k \le N,\ Nh = T\}$ with values in E equipped with the norm

$$\|\phi^\tau\|_{C_\tau(E)} = \max_{1\le k\le N} \|\phi(t_k)\|_E .$$

Let A be a strongly positive operator. With the help of A we introduce the fractional spaces $E_\alpha(E,A),\ 0<\alpha<1,$ consisting of all $v \in E$ for which the following norms are finite

$$\|v\|_\alpha = \|v\|_E + \sup_{\lambda>0} \|\lambda^\alpha A(\lambda + A)^{-1} v\|_E .$$

Throughout the article constants are indicated by $M(\alpha,\beta,\ldots)$ where the constant depends only on $\alpha,\beta,\ldots$. Then, the following theorem on well-posedness of problem (2) is established.

Theorem 1. For the solution problem (2), the following coercive stability estimates

$$\left\| \left\{ \frac{u_k^h - u_{k-1}^h}{\tau} \right\}_{k=1}^{N} \right\|_{C_\tau\left(\overset{\circ}{C}{}_h^{2\alpha}\right)} + \left\| \left\{ D_h^2 u_k^h \right\}_{k=1}^{N} \right\|_{C_\tau\left(\overset{\circ}{C}{}_h^{2\alpha}\right)}$$

$$\le M(q,s) \left\| \left\{ \frac{\rho(t_k) - \rho(t_{k-1})}{\tau} \right\}_{k=1}^{N} \right\|_{C[0,T]_\tau}$$

$$+M(\tilde{a},\phi,\alpha,T)\left(\left\|D_h^2\varphi^h\right\|_{\overset{\circ}{C}{}_h^{2\alpha}}+\left\|\left\{f^h(t_k)\right\}_{k=1}^N\right\|_{C_\tau\left(\overset{\circ}{C}{}_h^{2\alpha}\right)}+\left\|\rho^\tau\right\|_{C[0,T]_\tau}\right),$$

$$\left\|p^\tau\right\|_{C[0,T]_\tau}\leq M(q,s)\left\|\left\{\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right\}_{k=1}^N\right\|_{C[0,T]_\tau}$$

$$+M(\tilde{a},\phi,\alpha,T)\left(\left\|D_h^2\varphi^h\right\|_{\overset{\circ}{C}{}_h^{2\alpha}}+\left\|\left\{f^h(t_k)\right\}_{k=1}^N\right\|_{C_\tau\left(\overset{\circ}{C}{}_h^{2\alpha}\right)}+\left\|\rho^\tau\right\|_{C[0,T]_\tau}\right)$$

hold. Here, $f^h(t_k)=\left\{f(t_k,x_n)\right\}_{n=1}^{M-1}$, $\varphi^h=\left\{\varphi(x_n)\right\}_{n=1}^{M-1}$, $\rho^\tau=\left\{\rho(t_k)\right\}_{k=0}^N$, $D_h^2u^h=\left\{\frac{u_{n+1}-2u_n+u_{n-1}}{h^2}\right\}_{n=1}^{M-1}$ and $\tilde{a}=\frac{1}{q_s}(aD_h^2q^h-\sigma q^h)$.

The proof of theorem is based on the inequality

$$\left|p^k\right|\leq M(q,s)\left(\max_{1\leq k\leq N}\left|\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right|+\max_{1\leq k\leq N}\left|\frac{w_k^h-w_{k-1}^h}{\tau}\right|_{\overset{\circ}{C}{}_h^{2\alpha}}\right)$$

where $\left\{w_k^h\right\}_{k=0}^N$ is the solution of the following difference scheme

$$\begin{cases}\dfrac{w_n^k-w_n^{k-1}}{\tau}=a(x_n)\dfrac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}-a(x_n)\dfrac{\rho(t_k)-w_s^k}{q_s}\dfrac{q_{n+1}-2q_n+q_{n-1}}{h^2}\\ -\sigma\dfrac{\rho(t_k)-w_s^k}{q_s}q_n-\sigma w_n^k+f(t_k,x_n),\ x_n=nh,\ t_k=k\tau,\\ 1\leq k\leq N,\ 1\leq n\leq M-1,\ Mh=l,\ N\tau=T,\\ w_1^k-w_0^k=0,\ w_M^k-w_{M-1}^k=h\psi(t_k),\ 0\leq k\leq N,\\ w_n^0=\varphi(x_n),\ 0\leq n\leq M,\end{cases}\tag{3}$$

and the following theorems.

Theorem 2. The Following coercive stability estimate

$$\left\| \left\{ \frac{w_k^h - w_{k-1}^h}{\tau} \right\}_{k=1}^{N} \right\|_{C_\tau\left(\overset{\circ}{C}{}_h^{2\alpha}\right)} \le M(\tilde{a},\phi,\alpha,T)$$

$$\times \left(\left\| \varphi^h \right\|_{\overset{\circ}{C}{}_h^{2\alpha}} + \left\| \left\{ f^h(t_k) \right\}_{k=1}^{N} \right\|_{C_\tau\left(\overset{\circ}{C}{}_h^{2\alpha}\right)} + \left\| \rho^\tau \right\|_{C[0,T]_\tau} \right)$$

holds.

Theorem 3. For $0 < \alpha < \frac{1}{2}$ the norms of the spaces $E'_\alpha(C[0,l]_h,\ A_h^x)$ and $C^{2\alpha}[0,l]_h$ are equivalent.

3. NUMERICAL RESULTS

For the numerical verification of our algorithm, we assume that the diffusion coefficient $a = 1,$ $q(x) = 1$ and $f(t,x) = 0.$ We consider the following problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + p(t),\ \ x \in (0,\pi),\ \ t \in (0,1], \\ u(0,x) = \cos x + \dfrac{1}{4}x^4,\ \ x \in [0,\pi], \\ u_x(t,0) = 0,\ \ u_x(t,1) = 6t\pi + \pi^3,\ \ t \in [0,1], \\ u\left(t,\dfrac{1}{2}\right) = \rho(t) = e^{-t}\cos\dfrac{1}{2} + \dfrac{3}{4}t + \dfrac{1}{64},\ \ t \in [0,1]. \end{cases} \tag{4}$$

The exact solution of the given problem is $u(t,x) = e^{-t}\cos x + 3tx^2 + \frac{1}{4}x^4$ and of the control parameter $p(t)$ is $-6t.$

First, applying the Rothe difference scheme (2),

$$\begin{cases} \dfrac{u_n^k - u_n^{k-1}}{\tau} = \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + p^k, \ \ p^k = p(t_k), \ t_k = k\tau, \\ 1 \le k \le N, \ 1 \le n \le M-1, \ Mh = \pi, \ N\tau = 1, \\ u_n^0 = \cos(x_n) + \dfrac{1}{4}(x_n)^4, \ \ x_n = nh \ \ 0 \le n \le M, \\ u_0^k = u_1^k, \ \dfrac{u_{M-1}^k - u_M^k}{h} = 6t_k\pi + \pi^3, \ t_k = k\tau, \ 0 \le k \le N, \\ u_s^k = \rho(t_k) = \cos\left(\dfrac{1}{2}\right)e^{-t_k} + \dfrac{3}{4}t_k + \dfrac{1}{64}, \ t_k = k\tau, \ 0 \le k \le N, \ s = \left[\dfrac{1}{2h}\right] \end{cases} \tag{5}$$

is constructed.

We need to calculate the approximate value of the control parameter $p(t)$. The value of $p(t_k)$ at the grid points can be obtained from the equation

$$p^k = \frac{\rho(t_k) - \rho(t_{k-1})}{\tau} - \frac{w_s^k - w_s^{k-1}}{\tau}, \ 1 \le k \le N, \tag{6}$$

where $w_s^r, \ r = k, \ k-1$ is the solution of the difference scheme

$$\begin{cases} \dfrac{w_n^k - w_n^{k-1}}{\tau} = \dfrac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2}, \ 1 \le k \le N, \ 1 \le n \le M-1, \ Mh = \pi, \ N\tau = T, \\ w_0^k = w_1^k, \ \dfrac{w_{M-1}^k - w_m^k}{h} = 6t_k\pi + \pi^3, \ t_k = k\tau, \ 0 \le k \le N, \\ w_n^0 = \cos(x_n) + \dfrac{1}{4}(x_n)^4, \ 0 \le n \le M. \end{cases} \tag{7}$$

The difference scheme (7) can be arranged as

$$\begin{cases} \left(-\frac{1}{h^2}\right)w_{n+1}^k + \left(\frac{1}{\tau}+\frac{2}{h^2}\right)w_n^k + \left(-\frac{1}{h^2}\right)w_{n-1}^k + \left(-\frac{1}{\tau}\right)w_n^{k-1} = 0, \\ 1 \le k \le N,\ 1 \le n \le M-1,\ Mh = l,\ N\tau = T \\ w_0^k = w_1^k,\ \frac{w_{M-1}^k - w_m^k}{h} = 6t_k\pi + \pi^3,\ t_k = k\tau,\ 0 \le k \le N, \\ w_n^0 = \cos(x_n) + \frac{1}{4}(x_n)^4,\ 0 \le n \le M. \end{cases}$$

First, applying the first order of accuracy difference sceme (7), we obtain $(N+1)\times(M+1)$ system of linear equations and we write them in the matrix form

$$Aw^k + Bw^{k-1} = D\varphi^k, 1 \le k \le N, w^0 = \left\{\cos(x_n) + \tfrac{1}{4}(x_n)^4\right\}_{n=0}^{M} \tag{8}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & . & 0 & 0 & 0 \\ x & y & x & 0 & . & 0 & 0 & 0 \\ 0 & x & y & x & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & x & y & x \\ 0 & 0 & 0 & 0 & . & 0 & 1 & -1 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & v & 0 & . & 0 & 0 & 0 \\ 0 & 0 & v & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$$

Here,

$$x=-\tfrac{1}{h^{2}}, y=\tfrac{1}{\tau}+\tfrac{2}{h^{2}}+1, v=-\tfrac{1}{\tau},$$

$$w^{r}=\begin{bmatrix} w_0^r \\ \vdots \\ w_M^r \end{bmatrix}_{(M+1)\times 1} \qquad f \text{ or } r=k,k-1$$

$$\varphi^{k}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\left(6t_k\pi+\pi^3\right) \end{bmatrix}_{(M+1)\times 1}$$

and D is $(M+1)\times(M+1)$ identity matrix. Using (8), we can obtain that

$$w^{k}=A^{-1}\left(D\varphi^{k}-Bw^{k-1}\right), k=1,2,...,N, w^{0}=\left\{\cos\left(x_n\right)+\tfrac{1}{4}\left(x_n\right)^{4}\right\}_{n=0}^{M} \qquad (9)$$

Then, we can reach to the solution of $w_n^k, 0\le k\le N, 0\le n\le M$. Applying the first order of accuracy difference scheme (5) and (6), we have again $(N+1)\times(M+1)$ system of linear equations and we write them in the matrix form where

$$A_{2^{u^k}}+B_{2^{u^{k-1}}}=D_{\varphi^k}, 1\le k\le N, u^{\circ}=\left\{\cos\left(x_n\right)+\frac{1}{4}\left(x_n\right)^{4}\right\}_{n=0}^{M}$$

where

$$A_2=A, B_2=B$$

Here,

$$u^{r}=\begin{bmatrix} u_0^r \\ \vdots \\ u_M^r \end{bmatrix}_{(M+1)\times 1} \qquad f \text{ or } r=k,k-1,$$

$$\varphi^k = \begin{bmatrix} 0 \\ \phi_1^k \\ \vdots \\ \phi_{M-1}^k \\ h\left(6t_k\pi + \pi^3\right) \end{bmatrix}_{(M+1)\times 1}, \phi_n^k = p^k$$

To solve the resulting difference equations, we again apply the iterative method given in (9).

Now, we will give the results of the numerical analysis. The numerical solutions are recorded for different values of N and M and u_n^k represents the numerical solutions of these difference schemes at $\left(t_{k,}x_n\right)$.

Table 1 gives the relative error between the exact solution of $p(t)$ and the solutions derived by the numerical process. The error is computed by

$$E_p = \frac{\max\limits_{1\le k\le N}\left|p(t_k) - p_k\right|}{\max\limits_{1\le k\le N}\left|p(t_k)\right|}.$$

TABLE 1: Error analysis for $p(t)$.

	N=30	N=60	N=90
Rel. Error	0.0617	0.0286	0.0156

Table 2 gives the error analysis between the exact solution and the solutions derived by difference schemes. Table 2 constructed for $N = M = 30, 60$ and 90 respectively. For their comparison, the error is computed by

$$E = \frac{\max\limits_{\substack{1\le k\le N\\ 1\le n\le M}}\left|u(t_k, x_n) - u_n^k\right|}{\max\limits_{\substack{1\le k\le N\\ 1\le n\le M}}\left|u(t_k, x_n)\right|}.$$

TABLE 2: Error analysis for the exact solution $u(t,x)$.

Method	N=M=30	N=M=60	N=M=90
1^{st} order accuracy d.s	0.0401	0.0200	0.0130

The obtained results also show that the numerical solutions are stable and converge to the exact solution. A similar approach can be applied to general boundary conditions. High order accuracy difference scheme can be investigated by using the operator approaches.

REFERENCES

Ashyralyev, A. 2010. On a problem of determining the parameter of a parabolic equation. *Ukr. Math. J.* **9**: 1-11.

Borukhov, V. T. and Vabishchevich, P. N. 2000. Numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation. *Comput. Phys. Commun.* **126**: 32-36.

Cannon, J. R. and Yin, Hong-Ming. 1990. Numerical solutions of some parabolic inverse problems. *Numer. Meth. Part. D. E.* **2**: 177-191.

Cannon, J. R., Lin, Y. and Wang, S. 1991. Determination of a control parameter in a parabolic differential equation. *J. Austral. Math. Soc., Ser. B.* **33**: 149-163.

Choulli, M. and Yamamoto, M. 1996. Generic well-posedness of an inverse parabolic problem-the Hölder-space approach. *Inverse Probl.* **12**: 195-205.

Choulli, M. and Yamamoto, M. 1999. Generic well-posedness of a linear inverse parabolic problem with diffusion parameter. *J.Inv. Ill-Posed Problems.* **7**(3): 241-254.

Dehghan, M. 2003. Finding a control parameter in one-dimensional parabolic equations. *Appl. Math. Comput.* **135**: 491-503.

Ivanchov, N. I. 1995. On the determination of unknown source in the heat equation with nonlocal boundary conditions. *Ukr. Math. J.* **4**7(10): 1438-1441.

Liu, Fung-Bao. 2008. A modified genetic algorithm for solving the inverse heat transfer problem of estimating plan heat source. *Int. J. Heat Mass Tran.* **51**: 3745-3752.

Prilepko, A. I. and Kostin, A. B. 1992. Some inverse problems for parabolic equations with final and integral observation. *Mat. Sb.* **183**(4): 49-68.

Samarskii, A. A. and Vabishchevich, P. N. 2007. *Numerical methods for solving inverse problems of mathematical physics.* Inverse and Ill-posed Problems Series. Berlin, Newyork: Walter de Gruyter.

Yang, Ching-Yu. 1998. A sequential method to estimate the strength of the heat source based on symbolic computation. *Int. J. Heat Mass Tran.* **41**(14): 2245-2252.

Yan, L., Fu, Chu-Li and Yang, Feng-Lian. 2008. The method of fundamental solutions for the inverse heat source problem. *Eng. Anal. Bound. Elem.* **32**: 216-222.